

Entanglement between Two Uses of a Noisy Multipartite Quantum Channel Enables Perfect Transmission of Classical Information

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Suppose that m senders want to transmit classical information to n receivers with zero probability of error using a noisy multipartite communication channel. The senders are allowed to exchange classical, but not quantum, messages among themselves, and the same holds for the receivers. If the channel is classical, a single use can transmit information if and only if multiple uses can. In sharp contrast, we exhibit, for each m and n with $m \geq 2$ or $n \geq 2$, a quantum channel of which a single use is not able to transmit information yet two uses can. This latter property requires and is enabled by quantum entanglement.

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The maximum rate at which a communication channel carries information is characterized by a quantity called its *channel capacity*. Different constraints on the channel give rise to variants of channel capacity. For example, the most commonly studied capacity, referred to simply as the Shannon capacity, can be attained using encoding and decoding on a large number of uses of the channel, with the error probability approaching 0.

In 1956 Shannon introduced the notion of zero-error capacity to characterize the ability of noisy channels to transmit classical information with zero probability of error [1]. The study of this notion and the related topics has since then grown into a vast field called zero-error information theory [2], partly motivated by the fact that no error may be allowed in some applications, or the channel is not available for the large number of uses required for attaining a small error probability. Unlike Shannon capacity, the calculation of zero-error capacity is in essence a combinatorial optimization problem on graphs and may be extremely difficult even for very simple graphs. This difficulty has in turn stimulated a great deal of research in graph theory (e.g. [3]), a recent highlight of which is the resolution of the Strong Perfect Graph Conjecture [4].

Since the physical process underlying all communication channels, such as optical fibers, is quantum mechanical, a quantum information theory is necessary to capture the full potential of communication channels. While much progress has been made on understanding the quantum analogy of Shannon capacity, little is known about quantum zero-error capacity for transmitting classical information. Some basic facts on the latter subject were observed in Ref. [5], while it was shown that the capacity is in general also extremely difficult to compute [6].

The purpose of this Letter is to demonstrate that quantum zero-error capacity behaves dramatically different from the corresponding classical capacity, and the dif-

ference is due to the effect of quantum entanglement. We consider the following “multi-user” scenario: a set of m senders want to send classical information with zero probability of error to a set of n receivers through a noisy channel \mathcal{E} . We impose the following LOCC (Local Operations and Classical Communication) requirement: The senders are allowed to exchange classical, but not quantum, messages, and the same holds for the receivers. Note that if \mathcal{E} is a classical channel, the LOCC restriction does not reduce the capacity. However, when \mathcal{E} is a quantum channel, the LOCC requirement may reduce the capacity. Multi-user channels in which no communication is allowed within senders and receivers have been studied widely in classical information theory (see, e.g., Chapter 14 of Ref. [7]). Our definition of multi-user channels is a natural extension of such channels, and captures realistic settings where quantum communication is expensive. Multipartite quantum communication was studied in Ref. [8]. The scenario considered there differs from ours in several important aspects: in their model, (a) the quantum channel may be assisted with one- or two-way classical channels; (b) their purpose was to study the additivity of the channel capacity for transmitting *quantum* information with vanishing error probability, while our focus is on the capacity for transmitting classical information with zero-error probability.

We now describe our main result. When \mathcal{E} is classical, it is straightforward to see that its capacity is 0 if and only if one use of \mathcal{E} cannot transmit information. In sharp contrast, we show that this is not true for quantum channels in general. In particular, we construct a quantum channel \mathcal{E} , for each pair of m and n , $m \geq 2$ or $n \geq 2$, that one use of \mathcal{E} is not able to transmit information yet two uses can. The latter property can be achieved in two different ways: (1) The senders apply \mathcal{E} to create a maximally entangled state between the re-

ceivers. The receivers then distinguish the output states of the second use of \mathcal{E} by teleportation. (2) The senders locally prepare maximally entangled states between the two uses of \mathcal{E} . The effect of the second case cannot be observed under the assumptions of Refs. [5, 6] where only product input states between two uses are allowed. Fig. 1 demonstrates our construction for $m = 2$ and $n = 1$.

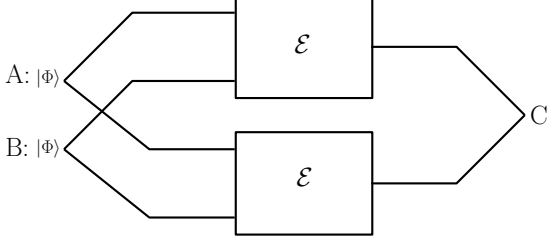


FIG. 1: \mathcal{E} is a noisy quantum channel from Alice and Bob to Charles. With separable input states, Alice and Bob cannot transmit classical information to Charles perfectly. Interestingly, by using \mathcal{E} twice, the senders can transmit a classical bit perfectly to the receiver. To do so, each sender first prepares a locally maximally entangled state $|\Phi\rangle = 1/2(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$. If Alice wants to send “0” to Charles, they simply apply \mathcal{E} twice. Instead, if Alice wants to send “1” to Charles, she first performs $U = |0\rangle\langle 0| - |1\rangle\langle 1| + |2\rangle\langle 2| - |3\rangle\langle 3|$ on one particle of her maximally entangled state and then the senders use \mathcal{E} two times. Bob can also send one single bit perfectly to Charles in a similar way.

Our construction in Case (2) uses the notion of completely entangled subspace, which has been studied by several authors recently [9, 10, 11, 12, 13, 14]. More precisely, for any $m > 1$, we construct a partition of an m -partite state space into two orthogonal subspaces, each of which contains no nonzero product state. Such partitions were found in Refs. [15, 16], and can be used to construct counterexamples to the additivity of minimum output p -Rényi entropy for p close to 0 [16]. Unfortunately, previous partitions are not sufficient for our purpose.

We define some notions necessary for describing our constructions. The set of n -bit binary strings is denoted by $\{0, 1\}^n$. Denote the complement of $x \in \{0, 1\}^n$ by \bar{x} . We associate each Hilbert space \mathcal{H} a fixed orthonormal basis, referred to as the computational basis and is usually denoted by $\{|i\rangle : 0 \leq i \leq \dim(\mathcal{H}) - 1\}$ or $\{|x\rangle : x \in \{0, 1\}^n\}$ when $\dim(\mathcal{H}) = 2^n$. The operator space on \mathcal{H} is denoted by $\mathcal{B}(\mathcal{H})$. If $R \in \mathcal{B}(\mathcal{H})$, denote by R^T its transpose with respect to the computational basis. Denote the application of an operator $U \in \mathcal{B}(\mathcal{H}_k)$ on the k 'th component of a multipartite system $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ by U^k .

Let $m, n \geq 1$ be positive integers. Let $\{A_1, \dots, A_m\}$ be a set of senders and $\{B_1, \dots, B_n\}$ be a set of receivers. Their state spaces are \mathcal{H}_{A_k} , $k = 1, \dots, m$, and \mathcal{H}_{B_ℓ} , $\ell = 1, \dots, n$, respectively. Denote by $\mathcal{H}_{\text{in}} = \otimes_{k=1}^m \mathcal{H}_{A_k}$ and $\mathcal{H}_{\text{out}} = \otimes_{k=1}^n \mathcal{H}_{B_k}$. An (m, n) multi-user quantum

channel \mathcal{E} is a completely positive trace-preserving map from $\mathcal{B}(\mathcal{H}_{\text{in}})$ to $\mathcal{B}(\mathcal{H}_{\text{out}})$, and is used as follows. The senders start with $|0\rangle \otimes \cdots \otimes |0\rangle$, and encode a message k into a state $\rho_k \in \mathcal{B}(\mathcal{H}_{\text{in}})$ through an LOCC protocol. The receivers receive $\mathcal{E}(\rho_k)$, and decode the message k by LOCC.

Define $\alpha_{\text{local}}(\mathcal{E})$ to be the maximum integer N with which there exist a set of states $\rho_1, \dots, \rho_N \in \mathcal{B}(\mathcal{H}_{\text{in}})$ such that: (a) Each ρ_k can be locally prepared by the senders, and (b) $\mathcal{E}(\rho_1), \dots, \mathcal{E}(\rho_N)$ can be perfectly distinguished by the receivers using LOCC. It follows from the linearity of superoperators that a set $\{\rho_k : k = 1, \dots, N\}$ achieving $\alpha_{\text{local}}(\mathcal{E})$ can be assumed without loss of generality to be orthogonal product pure states. Intuitively, one use of \mathcal{E} can be used to transmit $\log \alpha_{\text{local}}(\mathcal{E})$ bits of classical information perfectly. When $\alpha_{\text{local}}(\mathcal{E}) = 1$ it is clear that by a single use of \mathcal{E} the senders cannot transmit any classical information to the receivers perfectly.

The local zero-error classical capacity of \mathcal{E} , $C_{\text{local}}^{(0)}(\mathcal{E})$, is defined as follows:

$$C_{\text{local}}^{(0)}(\mathcal{E}) = \sup_{k \geq 1} \frac{\log_2 \alpha_{\text{local}}(\mathcal{E}^{\otimes k})}{k}. \quad (1)$$

Suppose that \mathcal{E} is classical (a so-called memoryless stationary channel), that is, $\mathcal{E} = \sum_k \langle k| \cdot |k\rangle \rho_k$ for some states ρ_k diagonalized under the computational basis $\{|k\rangle\}$. Then $\alpha_{\text{local}}(\mathcal{E}) = 1$ if and only if for all pairs of k and ℓ , $\rho_k \rho_\ell \neq 0$. Thus $\alpha_{\text{local}}(\mathcal{E}) = 1$ if and only if $\alpha_{\text{local}}(\mathcal{E}^{\otimes k}) = 1$ for any k . Therefore, $C_{\text{local}}^{(0)}(\mathcal{E}) = 0$ if and only if $\alpha_{\text{local}}(\mathcal{E}) = 1$. Our main theorem is:

Theorem 1. For any m and n with $m \geq 2$ or $n \geq 2$, there exists a multi-user channel \mathcal{E} from m senders to n receivers such that $\alpha_{\text{local}}(\mathcal{E}) = 1$ and $C_{\text{local}}^{(0)}(\mathcal{E}) > 0$.

Proof. For any positive integers m, n, m', n' , an (m, n) channel \mathcal{E} can be extended to an $(m + m', n + n')$ channel \mathcal{E}' , which ignores the inputs from the additional m' senders and provides $|0\rangle$ to all the additional n' receivers. Then for any k , $\alpha_{\text{local}}(\mathcal{E}^{\otimes k}) = \alpha_{\text{local}}(\mathcal{E}'^{\otimes k})$. Thus we need only to prove the theorem for $(m, n) = (1, 2)$ and $(2, 1)$.

Let $\rho_0 = |\alpha\rangle\langle\alpha|$ and $\rho_1 = \frac{1}{3}(I - \rho_0)$, where $|\alpha\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Consider the following $(1, 2)$ channel \mathcal{E}_{12} , which is from 1 qubit to 2 qubits:

$$\mathcal{E}_{12} = \langle 0| \cdot |0\rangle \rho_0 + \langle 1| \cdot |1\rangle \rho_1.$$

The only output pairs that are orthogonal are ρ_0 and ρ_1 . However, they cannot be distinguished by an LOCC protocol. Thus $\alpha_{\text{local}}(\mathcal{E}_{12}) = 1$. On the other hand, when \mathcal{E}_{12} is used twice, the two receivers, Alice and Bob, can distinguish the output states $\{\rho_0 \otimes \rho_0 = \mathcal{E}_{12}^{\otimes 2}(|00\rangle\langle 00|), \rho_0 \otimes \rho_1 = \mathcal{E}_{12}^{\otimes 2}(|01\rangle\langle 01|)\}$ by the following LOCC protocol: Alice first teleports her second qubit to Bob using the first qubit (which is maximally entangled with the first qubit

of Bob), then Bob applies a local measurement to distinguish ρ_0 and ρ_1 . Thus it follows from Eq. (1) that $C_{\text{local}}^{(0)}(\mathcal{E}_{12}) \geq 0.5$.

Now we turn to the construction of a $(2, 1)$ channel. The input space $\mathcal{H}_{\text{in}} = \mathcal{H}_A \otimes \mathcal{H}_B$, and the output space is \mathcal{H}_C . Each of \mathcal{H}_A and \mathcal{H}_B is a 4 dimensional space and \mathcal{H}_C is a qubit. Let $S_0 \subseteq \mathcal{H}_{\text{in}}$ be the subspace spanned by the following vectors:

$$\begin{aligned} |\psi_1\rangle &= |00\rangle - |11\rangle, \\ |\psi_2\rangle &= |22\rangle - |33\rangle, \\ |\psi_3\rangle &= |20\rangle - |31\rangle, \\ |\psi_4\rangle &= |02\rangle + |13\rangle, \\ |\psi_5\rangle &= |30\rangle - |03\rangle, \\ |\psi_6\rangle &= |10\rangle - \sqrt{2}|21\rangle + |32\rangle, \\ |\psi_7\rangle &= |01\rangle + \sqrt{2}|12\rangle + |23\rangle, \\ |\psi_8\rangle &= |10\rangle - |32\rangle - |01\rangle + |23\rangle, \end{aligned} \quad (2)$$

and $S_1 = S_0^\perp$. Let P_ℓ be the projector onto S_ℓ , $\ell = 0, 1$. The channel $\mathcal{E}_{21} : \mathcal{B}(\mathcal{H}_{\text{in}}) \rightarrow \mathcal{B}(\mathcal{H}_C)$ is defined as:

$$\mathcal{E}_{21}(\rho) = \text{tr}(P_0\rho)|0\rangle\langle 0| + \text{tr}(P_1\rho)|1\rangle\langle 1|. \quad (3)$$

Let $U = |0\rangle\langle 0| - |1\rangle\langle 1| + |2\rangle\langle 2| - |3\rangle\langle 3|$. The projections P_0 and P_1 have the following useful properties:

- (i) $P_\ell = P_\ell^T$ and $P_\ell = U^i P_\ell U^i$, for any $\ell \in \{0, 1\}$ and $i = A, B$. As a consequence, $P_\ell^T P_\ell = P_\ell^T U^i P_\ell = 0$. (Note that $U^\dagger = U$).
- (ii) Neither S_0 nor S_1 contains a product state, i.e. both S_0 and S_1 are completely entangled.

Property (i) can be verified by inspection. We now prove Property (ii). Let $|\Psi\rangle = (\sum_{k=0}^3 a_k |k\rangle)^A \otimes (\sum_{\ell=0}^3 b_\ell |\ell\rangle)^B$ be a product vector orthogonal to $|\psi_k\rangle$, $k = 1, \dots, 7$. Then

$$\begin{aligned} a_0 b_0 - a_1 b_1 &= 0, \\ a_2 b_2 - a_3 b_3 &= 0, \\ a_2 b_0 - a_3 b_1 &= 0, \\ a_0 b_2 + a_1 b_3 &= 0, \\ a_3 b_0 - a_0 b_3 &= 0, \\ a_1 b_0 - \sqrt{2} a_2 b_1 + a_3 b_2 &= 0, \\ a_0 b_1 + \sqrt{2} a_1 b_2 + a_2 b_3 &= 0. \end{aligned} \quad (4)$$

Suppose that $a_0 b_0 \neq 0$. Assume without loss of generality that $a_0 = b_0 = 1$. Then

$$\begin{aligned} a_1 b_1 &= 1, \quad a_2 b_2 = a_3 b_3, \quad a_2 = a_3 b_1, \\ b_2 &= -a_1 b_3, \quad a_3 = b_3, \quad b_1 = -\sqrt{2} a_1 b_2 - a_2 b_3. \end{aligned} \quad (5)$$

By $a_1 b_1 = 1$ and $a_2 = a_3 b_1$ we have $a_3 = a_1 a_2$. Substituting $b_2 = -a_1 b_3$ and $a_3 = b_3$ into $a_2 b_2 = a_3 b_3$ we have $-a_1 a_2 a_3 = a_3^2$. If $a_3 = 0$ then we have $b_2 = b_3 = 0$, which

together with $b_1 = -\sqrt{2} a_1 b_2 - a_2 b_3$ implies $b_1 = 0$. However, this is impossible as we have $a_1 b_1 = 1$. Thus $a_3 \neq 0$. This together with $-a_1 a_2 a_3 = a_3^2$ implies $a_3 = -a_1 a_2$. However, we have already shown that $a_3 = a_1 a_2$. Thus $a_3 = -a_3 = 0$, again a contradiction. Therefore $a_0 b_0 = 0$. Note that if $a_k b_\ell = 0$ and for a nonzero constant λ , $\lambda a_k b_{\ell'} = a_{k'} b_\ell$, then $a_k b_{\ell'} = a_{k'} b_\ell = 0$. Applying this inference rule many times, one concludes that all $a_k b_\ell = 0$, $0 \leq k, \ell \leq 3$ in both $a_0 = 0$ and $b_0 = 0$ cases. Thus $|\Psi\rangle = 0$, and S_1 contains no product state. By Property (i), this implies that S_0 does not contain a product state, either.

It follows from Property (ii) that $\mathcal{E}_{21}(|\psi\rangle\langle\psi|)$ is a mixed state for any product state $|\psi\rangle$. Thus $\alpha_{\text{local}}(\mathcal{E}_{21}) = 1$. We now consider using \mathcal{E}_{21} twice. Let $|\Phi\rangle = 1/2(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$. Define $|\Psi_0\rangle$ and $|\Psi_1\rangle$ as follows:

$$|\Psi_0\rangle = |\Phi\rangle^{AA'} \otimes |\Phi\rangle^{BB'} \quad \text{and} \quad |\Psi_{1,i}\rangle = U^i |\Psi_0\rangle,$$

where $i \in \{A, A', B, B'\}$. Note that for any operators R_1 and R_2 , $\langle\Psi_0|R_1^{AB} \otimes R_2^{A'B'}|\Psi_0\rangle = \text{tr}((R_1)^T R_2)$. Thus for any $\ell \in \{0, 1\}$, applying Property (i), we have

$$\langle\Psi_0|P_\ell^{AB} \otimes P_\ell^{A'B'}|\Psi_0\rangle = \text{tr}(P_\ell^T P_\ell) = 0,$$

and

$$\langle\Psi_{1,i}|P_\ell^{AB} \otimes P_\ell^{A'B'}|\Psi_{1,i}\rangle = \text{tr}(P_\ell^T U^i P_\ell U^i) = 0.$$

Thus $\mathcal{E}_{21}^{\otimes 2}(|\Psi_0\rangle\langle\Psi_0|)$ and $\mathcal{E}_{21}^{\otimes 2}(|\Psi_{1,i}\rangle\langle\Psi_{1,i}|)$ are orthogonal. This can also be verified by the following facts:

$$\begin{aligned} \mathcal{E}_{21}^{\otimes 2}(|\Psi_0\rangle\langle\Psi_0|) &= (|00\rangle\langle 00| + |11\rangle\langle 11|)/2, \\ \mathcal{E}_{21}^{\otimes 2}(|\Psi_{1,i}\rangle\langle\Psi_{1,i}|) &= (|01\rangle\langle 01| + |10\rangle\langle 10|)/2. \end{aligned}$$

Therefore $\alpha_{\text{local}}(\mathcal{E}_{21}^{\otimes 2}) \geq 2$, and $C_{\text{local}}^{(0)}(\mathcal{E}_{21}) \geq 0.5$. \blacksquare

The quantum channel \mathcal{E}_{21} constructed above has the following desirable property: when it is used twice, each sender is able to transmit one classical bit without leaking any information about this bit to the other sender. This is based on the fact that on $U^A|\Psi_0\rangle$ and $U^B|\Psi_0\rangle$, the channel outputs the same state orthogonal to the output on $|\Psi_0\rangle$. While tensoring \mathcal{E}_{21} with trivial channels does not preserve this property, we are able to construct a family of channels \mathcal{E}_{m1} , $m \geq 3$, that have this property. Each \mathcal{E}_{m1} is a channel from m qubits to 1 qubit defined in analogy to \mathcal{E}_{21} with the following set of base vectors for S_0 ($S_1 = S_0^\perp$):

$$|\psi_0\rangle = |0\rangle^{\otimes m} + |1\rangle^{\otimes m}, \quad \text{and},$$

$$|\psi_x\rangle = |0\rangle|x\rangle - |1\rangle|\bar{x}\rangle, \quad x \in \{0, 1\}^{m-1}, \quad x \neq 0^{m-1}.$$

The proof for $\alpha_{\text{local}}(\mathcal{E}_{m1}) = 1$ and $C_{\text{local}}^{(0)}(\mathcal{E}_{m1}) \geq 0.5$ is similar to that for \mathcal{E}_{21} , thus we leave it to the interested reader.

We note that if the “privacy” property is not required, the input dimension of \mathcal{E}_{21} can be reduced to $3 \otimes 4$ with the following set of base vectors for S_0 :

$$\begin{aligned} |\psi'_1\rangle &= |10\rangle - |21\rangle, \\ |\psi'_2\rangle &= |02\rangle + |13\rangle, \\ |\psi'_3\rangle &= |20\rangle - |03\rangle, \\ |\psi'_4\rangle &= |00\rangle - \sqrt{2}|11\rangle + |22\rangle, \\ |\psi'_5\rangle &= |01\rangle + \sqrt{2}|12\rangle + |23\rangle, \\ |\psi'_6\rangle &= |00\rangle - |22\rangle - |01\rangle + |23\rangle. \end{aligned} \quad (6)$$

Now only Bob (the party with the 4 dimensional state space) can transmit a private bit with two uses of the channel. This is simply due to the fact that both Properties (i) and (ii) still hold with the restriction that $i = B$ in Property (i). We omit the details of the proof as it is similar to that for \mathcal{E}_{21} .

Our construction of \mathcal{E}_{21} and the above variant has another application on the additivity of the minimum output p -Rényi entropy $S_{\min}^{(p)}$. For $0 \leq p \leq \infty$ and a quantum channel \mathcal{E} , $S_{\min}^{(p)}(\mathcal{E})$ is defined as

$$S_{\min}^{(p)}(\mathcal{E}) = \min_{\rho} \frac{1}{1-p} \log(\text{tr}(\mathcal{E}(\rho)^p)),$$

where ρ is a density operator, and at $p = 0, 1, +\infty$, the right hand side takes the limit. The *additivity problem* on $S_{\min}^{(p)}$ asks if

$$S_{\min}^{(p)}(\mathcal{E}_1 \otimes \mathcal{E}_2) = S_{\min}^{(p)}(\mathcal{E}_1) + S_{\min}^{(p)}(\mathcal{E}_2), \quad \forall \mathcal{E}_1, \mathcal{E}_2. \quad (7)$$

The case of $p = 1$ is a central open problem in quantum information theory [17]. This motivates the study of the same question for other values of p by many authors. It has been shown that $S_{\min}^{(p)}$ is not additive for any $p > 1$ [18]. Very recently Ref. [16] showed that $S_{\min}^{(p)}$ is not additive for p in a neighborhood of 0, by constructing a counterexample for $p = 0$. This is done by first constructing two completely entangled subspaces S and S' of a bipartite space so that $S \otimes S'$ is not completely entangled. Two channels \mathcal{E}_1 and \mathcal{E}_2 are then defined based on S and S' , respectively, through the Choi-Jamiołkowski isomorphism. It remains open if $S_{\min}^{(0)}$ is additive when $\mathcal{E}_1 = \mathcal{E}_2$. We answer this question negatively: setting both S and S' to be the S_0 in \mathcal{E}_{21} (and its variant) gives a channel \mathcal{E} so that $S_{\min}^{(0)}(\mathcal{E}^{\otimes 2}) < 2S_{\min}^{(0)}(\mathcal{E})$. We omit the details of the argument as it is similar to that in Ref. [16].

In conclusion, we have shown that for a broad class of multi-user quantum channels, of which the communication within the senders and receivers is restricted to be LOCC, a single use of the channel cannot be used to transmit classical information with zero probability of error, while multiple uses can. The latter property requires, and is a consequence of, quantum entanglement

between different uses, thus cannot be achieved by classical channels.

For all the channels we know, the LOCC restriction on encoding and decoding is necessary. It remains an intriguing open problem if such channels exist for a single sender and a single receiver. We do not know the exact values of $C_{\text{local}}^{(0)}$ for the constructed channels, despite the simplicity of their definitions. The difficulty in computing them is not unexpected given that $C_{\text{local}}^{(0)}$ generalizes the classical concept of zero-error capacity, which can be notoriously difficult to compute even for simple channels. Developing methods for estimating $C_{\text{local}}^{(0)}$ is thus of great theoretical interest, besides its obvious practical usefulness.

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